

INDUCTIVE LIMITS OF PROJECTIVE C^* -ALGEBRAS

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ABSTRACT. We show that a C^* -algebra is an inductive limit of projective C^* -algebras if and only if it has trivial shape, i.e., is shape equivalent to the zero C^* -algebra. In particular, every contractible C^* -algebra is an inductive limit of projectives, and one may assume that the connecting morphisms are surjective. Interestingly, an example of Dadarlat shows that trivial shape does not pass to full hereditary sub- C^* -algebra. It then follows that the same fails for projectivity.

To obtain these results, we develop criteria for inductive limit decompositions, and we discuss the relation with different concepts of approximation.

As main application of our findings we show that a C^* -algebra is (weakly) projective if and only if it is (weakly) semiprojective and has trivial shape. It follows that a C^* -algebra is projective if and only if it is contractible and semiprojective. This confirms a conjecture of Loring.

1. INTRODUCTION

Shape theory is a tool to study global properties of spaces. It was developed, since homotopy theory gives useful results only for spaces with good local behavior. Shape theory is a way of abstracting from the local behavior of a space, and focusing on its global behavior, its "shape".

One way of doing this, is to approximate a possibly badly-behaved space by nicer spaces, the building blocks. In the commutative world the building blocks are the so-called absolute neighborhood retracts (ANRs). The approximation is organized in an inverse limit structure, and instead of looking at the original space one studies an associated inverse system of ANRs.

After shape theory was successfully used to study (commutative) spaces, it was introduced to the study of noncommutative spaces (C^* -algebras) by Effros and Kaminker, [EK86], and short after developed to its modern form by Blackadar, [Bla85]. Shape theory works best when restricted to metrizable spaces, and similarly for noncommutative shape theory one restricts attention to separable C^* -algebras.

The building blocks of noncommutative shape theory are the semiprojective C^* -algebras, which are defined in analogy to ANRs. Since the category of commutative C^* -algebras is dual to the category of spaces, the approximation by an inverse system for spaces is turned

Date: January 20, 2013.

2000 Mathematics Subject Classification. Primary 46L05, 46L85, 46M10, ; Secondary 46M20, 54C56, 55P55 .

Key words and phrases. C^* -algebras, non-commutative shape theory, projectivity, contractible C^* -algebras.

This research was supported by the Marie Curie Research Training Network EU-NCG and by the Danish National Research Foundation through the Centre for Symmetry and Deformation.

to an approximation by an inductive system for C^* -algebras. Then, approximating a C^* -algebra by "nice" C^* -algebras means to write it as an inductive limit of semiprojective C^* -algebras.

A natural question is, whether there are enough building blocks to approximate every space. This holds in the commutative world, as every metric space is an inverse limit of ANRs. The analogue for C^* -algebras is still an open problem, asked first by Blackadar:

Question 1.1 (see [Bla85, 4.4]). Are all C^* -algebras inductive limits of semiprojective C^* -algebras?

In this paper we study the related question, which C^* -algebras are inductive limits of projective C^* -algebras. We obtain the following answer:

Theorem (see 4.4 for the complete result). *Let A be a separable C^* -algebra. Then the following are equivalent:*

- (a) *A is shape equivalent to 0*
- (e) *A is an inductive limit of projective C^* -algebras*

Since every contractible C^* -algebra has shape equivalent to 0, this also gives a positive answer to question 1.1 for a large class of C^* -algebras. We proceed as follows:

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In **section 2 (Preliminaries)**, we remind the reader of the basic notions of noncommutative shape theory, in particular the notion of (weak) semiprojectivity and (weak) projectivity.

In **section 3 (Approximation and criteria for inductive limits)**, we discuss different concepts of how a C^* -algebra can be "approximated" by other C^* -algebras, for instance as an inductive limit. If \mathcal{C} is a class of C^* -algebras, then an inductive limit of algebras in \mathcal{C} is called an \mathcal{AC} -algebra. We also suggest to use the formulation that A is " \mathcal{C} -like" if it can be approximated by sub- C^* -algebras from \mathcal{C} , see 3.2 and 3.4.

Building on a one-sided approximate intertwining argument, due to Elliott in [Ell93, 2.1, 2.3], see 3.5, we show the following criteria for inductive limits:

Theorem (see 3.9 and 3.12). *Let \mathcal{C} be a class of weakly semiprojective C^* -algebras. Then:*

- (1) *Every separable \mathcal{AC} -like C^* -algebra is already an \mathcal{AC} -algebra.*
- (2) *Every \mathcal{AAC} -algebra is already an \mathcal{AC} -algebra.*

In **section 4 (Trivial shape)** we study the class of C^* -algebras that are shape equivalent to the zero C^* -algebra 0. We show that these are exactly the C^* -algebras that are inductive limit of projective C^* -algebras, see 4.4. Moreover, one may assume that the connecting morphisms are surjective, since we show in 4.9 that every inductive system can be changed so that the connecting morphisms become surjective while the limit is unchanged. As an interesting corollary, we obtain:

Corollary (see 4.5). *Every separable, contractible C^* -algebra is an inductive limit of projective C^* -algebras.*

We discuss closure properties of the class of C^* -algebras with trivial shape, see 4.6. It follows from an example of Dadarlat that this class is not closed under full hereditary sub- C^* -algebras, see 4.11. We deduce the following:

Proposition (see 4.12). *Projectivity does not pass to full hereditary sub- C^* -algebras.*

In **section 5 (Relations among the classes of (weakly) (semi-)projective C^* -algebras)** we show the following results, which are the exact analogues of results in commutative shape theory:

Theorem (see 5.6). *Let A be a C^* -algebra. Then the following are equivalent:*

- (a) *A is (weakly) projective*
- (b) *A is (weakly) semiprojective and has trivial shape*

Corollary (see 5.7 for the complete result). *Let A be a C^* -algebra. Then the following are equivalent:*

- (a) *A is projective*
- (c) *A is semiprojective and contractible*

This confirms a conjecture of Loring.

2. PRELIMINARIES

By a morphism between C^* -algebras we mean a $*$ -homomorphism. All considered C^* -algebras are assumed to be separable. By ideals we mean closed, two-sided ideals. We use the symbol " \simeq " to denote homotopy equivalence, both for objects and morphism.

We use the following notations. For $\varepsilon > 0$, a subset F of a C^* -algebra A is said to be ε -**contained** in another subset G , denoted by $F \subset_\varepsilon G$, if for every $x \in F$ there exists some $y \in G$ such that $\|x - y\| < \varepsilon$.

Given two morphisms $\varphi, \psi: A \rightarrow B$ between C^* -algebras and a subset $F \subset A$ we say φ **and ψ agree on F** , denoted $\varphi =^F \psi$, if $\varphi(x) = \psi(x)$ for all $x \in F$. If moreover $\varepsilon > 0$ is given, then we say φ **and ψ agree on F up to ε** , denoted $\varphi =_\varepsilon^F \psi$, if $\|\varphi(x) - \psi(x)\| < \varepsilon$ for all $x \in F$.

We consider shape theory for separable C^* -algebra in the sense of Blackadar, see [Bla85]. We shortly recall the main notions, and we begin with a paragraph that is a shortened version of [ST11, 2.2]:

2.1 ((Weakly) (semi-)projective C^* -algebras, see [ST11, 2.2]).

A morphism $\varphi: A \rightarrow B$ is called **(weakly) projective** if for any C^* -algebra C and any morphism $\sigma: B \rightarrow C/J$ to some quotient (and $\varepsilon > 0$, and finite subset $F \subset A$), there exists a morphism $\psi: A \rightarrow C$ such that $\pi \circ \psi = \sigma \circ \varphi$ (resp. $\pi \circ \psi =_{\varepsilon}^F \sigma \circ \varphi$), where $\pi: C \rightarrow C/J$ is the quotient morphism. This means that the diagram on the right can be completed to commute (up to ε on F).

$$\begin{array}{ccccc} & & & C & \\ & & \nearrow \psi & \downarrow \pi & \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\sigma} & C/J \end{array}$$

A C^* -algebra A is called **(weakly) projective** if the identity morphism $\text{id}_A: A \rightarrow A$ is (weakly) projective.

A morphism $\varphi: A \rightarrow B$ is called **(weakly) semiprojective** if for any C^* -algebra C , any increasing sequence of ideals $J_1 \triangleleft J_2 \triangleleft \dots \triangleleft C$ and any morphism $\sigma: B \rightarrow C/\overline{\bigcup_k J_k}$ (and $\varepsilon > 0$, and finite subset $F \subset A$), there exist an index k and a morphism $\psi: A \rightarrow C/J_k$ such that $\pi_k \circ \psi = \sigma \circ \varphi$ (resp. $\pi_k \circ \psi =_{\varepsilon}^F \sigma \circ \varphi$), where $\pi_k: C/J_k \rightarrow C/\overline{\bigcup_k J_k}$ is the quotient morphism. This means that the diagram on the right can be completed to commute (up to ε on F).

$$\begin{array}{ccccc} & & & C & \\ & & & \downarrow & \\ & & & C/J_k & \\ & & \nearrow \psi & \downarrow \pi_k & \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\sigma} & C/\overline{\bigcup_k J_k} \end{array}$$

A C^* -algebra A is called **(weakly) semiprojective** if the identity morphism $\text{id}_A: A \rightarrow A$ is (weakly) semiprojective.

2.2 (Inductive systems). By an **inductive system** we mean a sequence A_1, A_2, \dots of C^* -algebras together with morphisms $\gamma_k: A_k \rightarrow A_{k+1}$ for each k . We will denote such a system by $\mathcal{A} = (A_k, \gamma_k)$. If $k < l$, then we let $\gamma_{l,k} := \gamma_{l-1} \circ \dots \circ \gamma_{k+1} \circ \gamma_k: A_k \rightarrow A_l$ denote the composition of connecting morphisms. By $\varinjlim \mathcal{A}$ or $\varinjlim A_k$ we denote the inductive limit of an inductive system, and by $\gamma_{\infty,k}: A_k \rightarrow \varinjlim A_k$ we denote the canonical morphism into the inductive limit.

2.3 (Shape systems). A **shape system** for A is an inductive system (A_k, γ_k) such that $A \cong \varinjlim A_k$ and such that the connecting morphisms $\gamma_k: A_k \rightarrow A_{k+1}$ are semiprojective. Blackadar, [Bla85, Theorem 4.3], shows that every separable C^* -algebra has a shape system consisting of finitely generated (f.g.) C^* -algebras.

Two inductive systems $\mathcal{A} = (A_k, \gamma_k)$ and $\mathcal{B} = (B_n, \theta_n)$ are called **(shape) equivalent**, denoted $\mathcal{A} \sim \mathcal{B}$, if there exists an increasing sequences of indices $k_1 < n_1 < k_2 < n_2 < \dots$ and morphisms $\alpha_i: A_{k_i} \rightarrow B_{n_i}$ and $\beta_i: B_{n_i} \rightarrow A_{k_{i+1}}$ such that $\beta_i \circ \alpha_i \simeq \gamma_{k_{i+1}, k_i}$ and $\alpha_{i+1} \circ \beta_i \simeq \theta_{n_{i+1}, n_i}$ for all i . The situation is shown in the following diagram, which commutes up to homotopy.

$$\begin{array}{ccccccc} A_{k_1} & \xrightarrow{\gamma_{k_2, k_1}} & A_{k_2} & \xrightarrow{\gamma_{k_3, k_2}} & A_{k_3} & \longrightarrow & \dots \longrightarrow A \\ & \searrow \alpha_1 & \nearrow \beta_1 & \searrow \alpha_2 & \nearrow \beta_2 & \searrow \alpha_3 & \\ & B_{n_1} & \xrightarrow{\theta_{n_2, n_1}} & B_{n_2} & \xrightarrow{\theta_{n_3, n_2}} & B_{n_3} & \longrightarrow \dots \longrightarrow B \end{array}$$

If we have α_i, β_i as above with only $\beta_i \circ \alpha_i \simeq \gamma_{k_{i+1}, k_i}$ for all i , then we say \mathcal{A} is **(shape) dominated** by \mathcal{B} , denoted $\mathcal{A} \lesssim \mathcal{B}$. Of course $\mathcal{A} \sim \mathcal{B}$ implies $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{B} \lesssim \mathcal{A}$, but the converse is false. Nevertheless \sim is an equivalence relation, and \lesssim is transitive.

Any two shape systems of a C^* -algebra are equivalent. Given two C^* -algebras A and B we say A is **shape equivalent** to B , denoted $A \sim_{Sh} B$, if they have some shape systems that are equivalent. We say A is **shape dominated** by B , denoted $A \lesssim_{Sh} B$, if some shape system of A is dominated by some shape system of B .

Shape is coarser than homotopy in the following sense: If A and B are homotopy equivalent (denoted $A \simeq B$), then $A \sim_{Sh} B$. Moreover, if A is homotopy dominated by B , then $A \lesssim_{Sh} B$.

Theorem 2.4 (see [Bla85, Theorem 3.1, 3.3], [EK86, 3.2]). *Let $\varphi: A \rightarrow B$ be a semiprojective morphism, and (C_k, γ_k) an inductive system with limit C . Then:*

- (1) *Let $\sigma: B \rightarrow C$ be a morphism. Then for k large enough there exist morphisms $\psi_k: A \rightarrow C_k$ such that $\gamma_{\infty, k} \circ \psi_k \simeq \sigma \circ \varphi$ and such that $\gamma_{\infty, k} \circ \psi_k$ converges pointwise to $\sigma \circ \varphi$. This means that the diagram on the right can be completed to commute up to homotopy.*
- (2) *Let $\sigma_1, \sigma_2: B \rightarrow C_k$ be two morphisms with $\gamma_{\infty, k} \circ \sigma_1 \simeq \gamma_{\infty, k} \circ \sigma_2$. Then for $n \geq k$ large enough, already the morphisms $\gamma_{n, k} \circ \sigma_1 \circ \varphi$ and $\gamma_{n, k} \circ \sigma_2 \circ \varphi$ are homotopic. The situation is shown in the diagram on the right.*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \psi_k \downarrow \cdots & & \downarrow \sigma \\ C_k & \xrightarrow{\gamma_{\infty, k}} & C \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & & \\ \sigma_2 \swarrow & & \searrow \sigma_1 & & \\ C_k & \xrightarrow{\gamma_{n, k}} & C_n & \longrightarrow & C \end{array}$$

Remark 2.5. Let us see what the above theorem means for a semiprojective C^* -algebra A . Let (C_k, γ_k) be an inductive system with limit C . Consider the homotopy classes of morphisms from A to C_k , denoted by $[A, C_k]$. The connecting morphism $\gamma_k: C_k \rightarrow C_{k+1}$ induces a map $(\gamma_k)_*: [A, C_k] \rightarrow [A, C_{k+1}]$, and the morphism $\gamma_{\infty, k}: C_k \rightarrow C$ induces a map $(\gamma_{\infty, k})_*: [A, C_k] \rightarrow [A, C]$.

Note that $(\gamma_{\infty, k})_* = (\gamma_{\infty, k+1})_* \circ (\gamma_k)_*$, so that we get a natural map

$$\Phi: \varinjlim [A, C_k] \rightarrow [A, \varinjlim C_k] = [A, C].$$

Statement (1) of the above theorem 2.4 means that Φ is surjective, while statement (2) means exactly that Φ is injective.

The above theorem is proved using a mapping telescope construction, due to Brown. The same proof gives the following partial analogue of the above result for weakly semiprojective morphisms:

Proposition 2.6. *Let $\varphi: A \rightarrow B$ be a weakly semiprojective morphism, and (C_k, γ_k) an inductive system with limit C . Let further be given a morphism $\sigma: B \rightarrow C$, $\varepsilon > 0$ and a finite set $F \subset A$. Then there exists an index k and a morphism $\psi_k: A \rightarrow C_k$ such that $\gamma_{\infty, k} \circ \psi_k \stackrel{F}{=} \sigma \circ \varphi$.*

Remark 2.7 (Definition of weakly semiprojective C^* -algebras). For the definition of (weak) semiprojectivity of a morphism or a C^* -algebra one considers morphisms into a quotient C/J with $J = \overline{\bigcup_k J_k}$ and requires the existence of an (approximate) lift into C/J_k , see 2.1. Note that the C^* -algebras C/J_k form an inductive system with inductive limit C/J . The connecting morphisms $C/J_k \rightarrow C/J_{k+1}$ are quotient morphisms and therefore surjective.

Let us see that every inductive limit (D_k, γ_k) with surjective connecting morphisms $\gamma_k: D_k \rightarrow D_{k+1}$ is of the above form. Just set $C := D_1$ and $J_k := \ker(\gamma_{k,1})$. Then $C/J_k \cong D_k$, and $C/J \cong \varinjlim_k D_k$ where $J := \overline{\bigcup_k J_k} = \ker(\gamma_{\infty,1})$.

Thus, for the definition of (weak) semiprojectivity, one considers morphisms into inductive limits with surjective connecting morphisms. If one considers morphisms into a general inductive limit, then one only gets an approximate lift, see 2.4.

However, in the definition of weak semiprojectivity one only asks for an approximate lift anyway. As noted in 2.6, one gets such an approximate lift for a morphism into any inductive limit (also with not necessarily surjective connecting morphisms).

$$\begin{array}{ccccc}
 & & C/J_k & \cong & D_k \\
 & & \downarrow & & \downarrow \gamma_k \\
 & & C/J_{k+1} & \cong & D_{k+1} \\
 & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
 A & \xrightarrow{\varphi} & B & \xrightarrow{\sigma} & C/\overline{\bigcup_k J_k} \cong \varinjlim D_k
 \end{array}$$

ψ (dotted arrow from A to C/J_{k+1})

2.8 (Generators for C^* -algebras). Let A be a C^* -algebra. A subset $S \subset A_{\text{sa}}$ of self-adjoint elements is said to generate A , denoted $A = C^*(S)$, if A is the smallest sub- C^* -algebra of A containing S . The **generating rank** for A , denoted by $\text{gen}(A)$, is the smallest number $n \in \{1, 2, 3, \dots, \infty\}$ such that A contains a generating set of cardinality n .

Note that the generators are assumed to be self-adjoint. If g, h are two self-adjoint elements, then $\{g, h\}$ generates the same sub- C^* -algebra as the element $g + ih$. That is why a C^* -algebra is said to be singly generated if $\text{gen}(A) \leq 2$.

For more details on the generation rank and its behaviour with respect to operations, we refer the reader to Nagisa, [Nag].

Remark 2.9 (Finitely generated = finitely presented). While it is rather clear what it means that a C^* -algebra is finitely generated, it is not so obvious what it should mean that it is finitely presented. To speak of finite presentation, one needs a theory of universal C^* -algebras defined by generators and relations.

Depending on which relations one admits, one gets different notions of finite presentability. In [Bla85] for instance, only polynomial relations are considered. With this notion, not every finitely generated C^* -algebra is also finitely presented.

A more general concept is to understand by a relation any subset R of the universal C^* -algebra generated by a countable number of contractions

$$\mathcal{F}_\infty := C^*(x_1, x_2, \dots \mid \|x_i\| \leq 1).$$

This concept is for instance used in [Lor97], and it is flexible enough to show that every finitely generated C^* -algebra is already finitely presented, see [ELP98, Lemma 2.2.5].

Thus, in the results of [Lor97] we may always replace the assumption of finite presentation by finite generation, for example [Lor97, Lemma 15.2.1, p.118], see 3.8, and [Lor97, Lemma 15.2.2, p.119], see 3.10.

3. APPROXIMATION AND CRITERIA FOR INDUCTIVE LIMITS

In this section we will give criteria that allow one to write a C^* -algebra A as an inductive limit of other C^* -algebras that approximate A in a nice way. We start by reviewing the various ways a C^* -algebra can be "approximated" by other C^* -algebras, see 3.1. If \mathcal{C} is a class of C^* -algebras, then an inductive limit of algebras in \mathcal{C} is called an \mathcal{AC} -algebra. We suggest to use the formulation that A is " \mathcal{C} -like" if it can be approximated by sub- C^* -algebras from the class \mathcal{C} , see 3.2 and 3.4.

As basic tool to build an inductive limit decomposition we use one-sided approximate intertwining, see 3.5. These were introduced by Elliott in [Ell93, 2.1, 2.3] and they turned out to be very important in the classification of C^* -algebras, see also chapter 2.3 of Rørdam's book, [Rø02].

Assuming that the class \mathcal{C} consists of weakly semiprojective C^* -algebras, we deduce other criteria for inductive limits. In particular, every \mathcal{AC} -like C^* -algebra is an \mathcal{AC} -algebra, see 3.9, and every \mathcal{AAC} -algebra is already an \mathcal{AC} -algebra, see 3.12. The latter statement gives a criterion when an "inductive limit of inductive limits is an inductive limit".

For example, let \mathcal{C} be the class of finite direct sums of matrices over the circle algebra $C(\mathbb{T})$. Then the mentioned result means that an inductive limit of AT -algebras is itself an AT -algebra. This is a well-known result, see e.g. [LR95, Proposition 2] which is based on [Ell93, Theorem 4.3].

We are aware that many results in this section are known to the experts and special cases of the results have appeared in the literature, but we think it is useful to include this systematic treatment of criteria for inductive limits.

3.1 (Approximation). The term "approximation" is used in various contexts. For instance, if \mathcal{P} is some property that C^* -algebras might enjoy, then a C^* -algebra is usually called **approximately** \mathcal{P} , or \mathcal{AP} -algebra, if it can be written as an inductive limit of C^* -algebras with property \mathcal{P} . In this sense one speaks of "approximately homogeneous" and "approximately subhomogeneous" C^* -algebras.

Another concept is approximation by subalgebras. Given a C^* -algebra A , a family \mathcal{B} of sub- C^* -algebras is said to **approximate** A if for every finite subset $F \subset A$ and $\varepsilon > 0$ there exists some algebra $B \in \mathcal{B}$ such that $F \subset_\varepsilon B$. In the literature there appears also the terminology " \mathcal{B} locally approximates A ". Similarly, if \mathcal{P} is some property of C^* -algebras, then a C^* -algebra A that can be approximated by sub- C^* -algebras with property \mathcal{P} is sometimes called "locally \mathcal{P} ". In this sense one speaks of "locally (sub)homogeneous" C^* -algebras.

However, sometimes the word "local" might lead to confusion: Consider for instance the property of being contractible. We will show below, see 4.7, that a C^* -algebra has trivial shape if it is approximated by contractible sub- C^* -algebras. One would probably not like to phrase this as "locally contractible C^* -algebras have trivial shape", as this would

be in contradiction with the terminology used for commutative spaces. Many locally contractible¹ spaces have non-trivial shape.

We suggest the following definition:

Definition 3.2. *If \mathcal{P} is some property that C^* -algebras might enjoy, then a C^* -algebra is called \mathcal{P} -like if it can be approximated by sub- C^* -algebras with property \mathcal{P} .*

Remark 3.3 (\mathcal{P} -likeness). Using the above definition, the result 4.7 would read as: "A contractible-like C^* -algebra has trivial shape". This might sound cumbersome, but it is motivated by the concept of \mathcal{P} -likeness for commutative spaces, as defined in [MS63, Definition 1] and further developed in [MM92]. In the next result 3.4 we will show that for commutative C^* -algebras both concepts agree.

We are working in the category of pointed spaces and pointed maps since it is the natural setting to study non-unital commutative C^* -algebras, as pointed out in [Bla06, II.2.2.7, p.61]. If we include basepoints, then the definition of \mathcal{P} -likeness from [MS63] becomes: Let \mathcal{P} be a non-empty class of metric, pointed spaces. A compact, metric, pointed space (X, x_∞) is said to be \mathcal{P} -like if for every $\varepsilon > 0$ there exists a pointed map $f: X \rightarrow Y$ onto some $Y \in \mathcal{P}$ such that the sets $f^{-1}(y)$ have diameter $< \varepsilon$ (for all $y \in Y$).

One can show that (X, x_∞) is \mathcal{P} -like if and only if for every $\mathcal{U} \in \text{Cov}(X)$ there exists a (pointed) map $f: X \rightarrow Y$ onto some $Y \in \mathcal{P}$ and $\mathcal{V} \in \text{Cov}(Y)$ such that $f^{-1}(\mathcal{V}) \leq \mathcal{U}$. Here $\text{Cov}(X)$ denotes the family of normal, open covers of X , and we write $\mathcal{U}_1 \leq \mathcal{U}_2$ if the cover \mathcal{U}_1 refines the cover \mathcal{U}_2 (see chapter 2 of Nagami's book [Nag70] for definitions and further explanations). This equivalent formulation was used to generalize the notion of \mathcal{P} -likeness to non-compact spaces, see [MM92].

Note that we have used \mathcal{P} to denote both a class of spaces and a property that spaces might enjoy. These are just different viewpoints, as we can naturally assign to a property the class of spaces with that property, and vice versa to each class of spaces the property of lying in that class.

Let us use the following notation for the next result: If (X, x_∞) is a pointed space, then $C_0(X, x_\infty) = \{a: X \rightarrow \mathbb{C} \mid a(x_\infty) = 0\}$ denotes the C^* -algebra of continuous functions on X vanishing at the basepoint.

Proposition 3.4. *Let X be a compact, metric space with a basepoint $x_\infty \in X$, and let \mathcal{P} be a class of pointed, compact, metric spaces. Then the following are equivalent:*

- (a) (X, x_∞) is \mathcal{P} -like
- (b) $C_0(X, x_\infty)$ is approximated by sub- C^* -algebras $C_0(Y, y_\infty)$ with $(Y, y_\infty) \in \mathcal{P}$

Proof. "(a) \Rightarrow (b)": We denote the metric of X by d . Let $a_1, \dots, a_k \in C_0(X, x_\infty)$, and $\varepsilon > 0$ be given. We need to find a sub- C^* -algebra $C_0(Y, y_\infty) \subset C_0(X, x_\infty)$ that contains the functions a_i up to ε .

¹A space X is called locally contractible if for each point $x \in X$ and every neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and V is contractible (in itself).

Since the a_i are absolutely continuous, we may find $\delta > 0$ such that $d(x, x') < \delta$ implies $\|a_i(x) - a_i(x')\| < \varepsilon$ for all i . From (a) we get a pointed space $(Y, y_\infty) \in \mathcal{P}$ and a pointed, surjective map $f: (X, x_\infty) \rightarrow (Y, y_\infty)$ together with a finite cover $\mathcal{V} = \{V_\alpha\} \in \text{Cov}(Y)$ such that the sets in $f^{-1}(\mathcal{V})$ have diameter $< \delta$.

Choose a partition of unity $\{e_\alpha\}$ in (Y, y_∞) that is subordinate to \mathcal{V} . Choose points $x_\alpha \in f^{-1}(V_\alpha)$ such that $x_\alpha = x_\infty$ if $y_\infty \in V_\alpha$. For each i define an element $b_i \in C_0(Y, y_\infty)$ as follows:

$$b_i := \sum_{\alpha} a_i(x_\alpha) e_\alpha$$

Note that $b_i(y_\infty) = 0$ since $e_\alpha(y_\infty)$ is non-zero only if $y_\infty \in V_\alpha$, but then $x_\alpha = x_\infty$.

For $x \in X$ we compute:

$$\begin{aligned} & \|a_i(x) - f^*(b_i)(x)\| \\ &= \|a_i(x) - \sum_{\alpha} a_i(x_\alpha) e_\alpha(f(x))\| \\ &= \left\| \sum_{\alpha} (a_i(x) - a_i(x_\alpha)) e_\alpha(f(x)) \right\| \quad \left[\text{since } \sum_{\alpha} e_\alpha = 1 \right] \\ &\leq \varepsilon \left\| \sum_{\alpha} e_\alpha(f(x)) \right\| \quad \left[\begin{array}{l} e_\alpha(f(x)) \neq 0 \text{ only if } f(x) \in V_\alpha \text{ but then} \\ d(x, x_\alpha) < \delta, \text{ hence } \|a_i(x) - a_i(x_\alpha)\| < \varepsilon \end{array} \right] \\ &= \varepsilon \end{aligned}$$

If we consider $C_0(Y, y_\infty)$ embedded into $C_0(X, x_\infty)$ via f^* , then we have just shown $a_i \in_\varepsilon C_0(Y, y_\infty)$.

"(b) \Rightarrow (a)": Let $\mathcal{U} = \{U_\alpha\} \in \text{Cov}(X)$ be a finite cover of X . By passing to a refinement, we may assume that x_∞ is contained in just one U_α , call it U_∞ . Since X is a normal space, we may find open sets $V_\alpha \subset X$ such that $V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$ and such that $\{V_\alpha\}$ is a cover of X . By Urysohn's lemma, there are continuous functions $a_\alpha: X \rightarrow \mathbb{C}$ that are 1 on $\overline{V_\alpha}$ and zero on $X \setminus U_\alpha$. Note that a_α vanishes on x_∞ for $\alpha \neq \infty$, so that $a_\alpha \in C_0(X, x_\infty)$ for $\alpha \neq \infty$.

From (b) we get a sub- C^* -algebra $C_0(Y, y_\infty)$ of $C_0(X, x_\infty)$ that contains the a_α ($\alpha \neq \infty$) up to $1/2$ and such that $(Y, y_\infty) \in \mathcal{P}$. The embedding corresponds to a pointed, surjective map $f: (X, x_\infty) \rightarrow (Y, y_\infty)$. For $\alpha \neq \infty$, let $b_\alpha \in C_0(Y, y_\infty)$ be elements such that $\|a_\alpha - f^*(b_\alpha)\| < 1/2$.

Define sets $W_\alpha \subset Y$ via:

$$W_\alpha := \{y \in Y \mid \|b_\alpha(y)\| > 1/2\} \quad (\text{for } \alpha \neq \infty)$$

$$W_\infty := Y \setminus f\left(\bigcup_{\alpha \neq \infty} \overline{V_\alpha}\right)$$

We compute:

$$\begin{aligned}
 f^{-1}(W_\alpha) &= \{x \in X \mid \|b_\alpha(f(x))\| \geq 1/2\} & (\text{for } \alpha \neq \infty) \\
 &\subset \{x \in X \mid \|a_\alpha(x)\| > 0\} \subset U_\alpha \\
 f^{-1}(W_\infty) &\subset X \setminus \bigcup_{\alpha \neq \infty} \overline{V_\alpha} \subset U_\infty \\
 f^{-1}(W_\alpha) &\supset \{x \in X \mid \|a_\alpha(x)\| \geq 1\} \supset \overline{V_\alpha} & (\text{for } \alpha \neq \infty)
 \end{aligned}$$

It follows $f^{-1}(\bigcup_{\alpha \neq \infty} W_\alpha) \supset \bigcup_{\alpha \neq \infty} \overline{V_\alpha}$, and so $\bigcup_{\alpha \neq \infty} W_\alpha \supset f(\bigcup_{\alpha \neq \infty} \overline{V_\alpha}) = Y \setminus W_\infty$. This shows that $\mathcal{W} := \{W_\alpha\}$ is a cover of Y and that $f^{-1}(\mathcal{W}) \leq \mathcal{U}$, as desired. \square

The following result formalizes the construction of a (special) one-sided approximate intertwining. The idea goes back to Elliott, [Ell93, 2.3,2.4], see also chapter 2.3 of Rørdam's book, [Rø02].

Proposition 3.5 (One-sided approximate intertwining). *Let A be a separable C^* -algebra, and A_i ($i \in I$) a collection of separable C^* -algebras together with morphisms $\varphi_i: A_i \rightarrow A$.*

Assume that the following holds: For every index $i \in I$, and $\varepsilon > 0$, and for every finite subsets $F \subset A_i$, $E \subset \ker(\varphi_i)$ and $H \subset A$, there exists some index $j \in I$ and a morphism $\psi: A_i \rightarrow A_j$ such that:

- (A1) $\varphi_j \circ \psi =_F^\varepsilon \varphi_i$
- (A2) $\psi =_E^\varepsilon 0$
- (A3) $H \subset_\varepsilon \text{im}(\varphi_j) = \varphi_j(A_j) \subset A$

Then A is isomorphic to an inductive limit of some of the algebras A_i . More precisely, there exist indices $i(1), i(2), \dots \in I$ and morphisms $\psi_k: A_{i(k)} \rightarrow A_{i(k+1)}$ such that $A \cong \varinjlim_k (A_{i(k)}, \psi_k)$.

Proof. By induction, we will construct a one-sided approximate intertwining as shown in the following diagram. This diagram does not commute, but it "approximately commutes".

$$\begin{array}{ccccccc}
 A_{i(1)} & \xrightarrow{\psi_1} & A_{i(2)} & \xrightarrow{\psi_2} & A_{i(2)} & \longrightarrow & \dots \longrightarrow B \\
 \varphi_{i(1)} \downarrow & & \varphi_{i(2)} \downarrow & & \varphi_{i(3)} \downarrow & & \downarrow \omega \\
 A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \dots \longrightarrow A
 \end{array}$$

Property (A1) is the essential requirement for constructing the one-sided approximate intertwining, i.e., to align some of the algebras A_k into an inductive system with limit B together with a canonical morphism $\omega: \varinjlim B \rightarrow A$. Property (A2) is used to get ω injective, and (A3) is used to ensure ω is surjective.

More precisely, we proceed as follows: Let $\{x_1, x_2, \dots\} \subset A$ be a dense sequence in A with $x_1 = 0$. We will construct the following:

- indices $i(1), i(2), \dots$
- morphisms $\psi_k: A_{i(k)} \rightarrow A_{i(k+1)}$
- finite subsets $F_k^1 \subset F_k^2 \subset \dots \subset A_{i(k)}$
- finite sets $E'_k \subset \ker(\varphi_{i(k)})$

such that:

- (a) $\psi_k(F_k^l) \subset F_{k+1}^l$ (for all $k, l \geq 1$)
- (b) $\bigcup_l F_k^l$ is dense in $A_{i(k)}$ (for each k)
- (c) E'_k contains $E_k := \{x \in F_k^k : \|\varphi_{i(k)}(x)\| < 1/2^{k-1}\}$ up to $1/2^{k-1}$ (for each k)
- (d) $\varphi_{i(k+1)} \circ \psi_k =_{1/2^k}^{F_k^k} \varphi_{i(k)}$
- (e) $\psi_k =_{1/2^k}^{E'_k} 0$
- (f) $\{x_1, \dots, x_k\} \subset_{1/2^k} \varphi_{i(k)}(F_k^k)$

Let us start with any $i(1)$, e.g. $i(1) = 1$. Since $x_1 = 0$, (f) is satisfied. We may find sets F_1^i and E'_1 to fulfill properties (a), (b) and (c).

Let us manufacture the induction step from k to $k+1$. We consider the index $i(k)$, the tolerance $1/2^{k+1}$, and the finite sets $F_k^k \subset A_{i(k)}$, $E'_k \subset \ker(\varphi_{i(k)})$, and $\{x_1, \dots, x_{k+1}\} \subset A$. By assumption, there is an index $i(k+1)$, and a morphism $\psi_k: A_{i(k)} \rightarrow A_{i(k+1)}$ satisfying conditions (d), (e) and (f). Then construct sets F_{k+1}^l and E'_{k+1} to fulfill properties (a), (b) and (c).

Set $B := \varinjlim_k (A_{i(k)}, \psi_k)$. We want to define morphisms $\omega_k: A_{i(k)} \rightarrow A$ as

$$\omega_k(a) := \lim_s \varphi_{i(s)} \circ \psi_{s,k}(a)$$

For this to make sense, we need to check that $\varphi_{i(s)} \circ \psi_{s,k}(a)$ is a Cauchy sequence (when running over s) for all $a \in A_{i(k)}$. It is enough to check it for a dense set. By property (f), $\bigcup_l F_k^l$ is dense in $A_{i(k)}$. Fix some $a \in F_k^l$. Then $\psi_{s,k}(a) \in F_s^s$ by (a) for all $s \geq k$. The following computation shows that the sequence $\varphi_{i(s)} \circ \psi_{s,k}(a)$ is Cauchy (for $r > s \geq k$):

$$\begin{aligned} & \|\varphi_{i(r)} \circ \psi_{r,k}(a) - \varphi_{i(s)} \circ \psi_{s,k}(a)\| \\ &= \left\| \sum_{t=s}^{r-1} (\varphi_{i(t+1)} \circ \psi_{t+1,k}(a) - \varphi_{i(t)} \circ \psi_{t,k}(a)) \right\| \\ &\leq \sum_{t=s}^{r-1} \|(\varphi_{i(t+1)} \circ \psi_t - \varphi_{i(t)})(\psi_{t,k}(a))\| \\ &\leq \sum_{t=s}^{r-1} 1/2^t \quad [\text{by (d), using } \psi_{t,k}(a) \in F_t^t] \\ &\leq 1/2^{s-1}. \end{aligned}$$

Note that $\omega_l \circ \psi_{l,k} = \omega_k$ for any $l \leq jk$. Thus, the morphisms ω_k fit together to define a morphism $\omega: B \rightarrow A$.

Injectivity of ω : Given any k , we first consider an element $a \in F_k^k$, and we compute:

$$\begin{aligned} \|\omega_k(a) - \varphi_{i(k)}(a)\| &= \left\| \lim_s \varphi_{i(s)} \circ \psi_{s,k}(a) - \varphi_{i(k)}(a) \right\| \\ &\leq \sum_{s \geq k} \|\varphi_{i(s+1)} \circ \psi_{s+1,k}(a) - \varphi_{i(s)} \circ \psi_{s,k}(a)\| \\ &< \sum_{s \geq k} 1/2^s \leq 1/2^{k-1} \end{aligned}$$

The construction was made in such a way that we can distinguish two different cases:

- Case 1: $\|\varphi_{i(k)}(a)\| \geq 1/2^{k-1}$. In that case $\omega_k(a) \neq 0$, since above we computed $\|\omega_k(a) - \varphi_{i(k)}(a)\| < 1/2^{k-1}$.
- Case 2: $\|\varphi_{i(k)}(a)\| < 1/2^{k-1}$. In that case, by (c), there exists some $e \in E'_k$ with $\|a - e\| < 1/2^{k-1}$. From (e) we get $\|\psi_k(e)\| < 1/2^k$. We compute: $\|\psi_k(a)\| = \|\psi_k(a - e + e)\| \leq 1/2^{i-k} + 1/2^k \leq 1/2^{k-2}$

This means: either the given $a \in F_k^k$ has non-zero image in A under the morphism ω_k , or otherwise it has a small image in B under the morphism $\psi_{\infty,k} = \psi_{\infty,k+1} \circ \psi_k$. By considering $\psi_{l,k}(a) \in F_l^l$ for all $l \geq k$ we derive that either $\omega_k(a) = \omega_l(\psi_{l,k}(a)) \neq 0$ or $\|\psi_{\infty,k}(a)\| = \|\psi_{\infty,l} \circ \psi_{l,k}(a)\| \leq 1/2^{k-2}$ for all $k \geq l$. We get that for any $a \in F_k^k$ we have $a \in \ker(\omega_k)$ if and only if $a \in \ker(\psi_{\infty,k})$.

Next we consider $a \in F_k^l$ for $l \geq k$. Then with $b := \psi_{l,k}(a) \in F_l^l$ we deduce:

$$\begin{aligned} a \in \ker(\omega_k) &\Leftrightarrow b \in \ker(\omega_l) \\ &\Leftrightarrow b \in \ker(\psi_{\infty,l}) \\ &\Leftrightarrow a \in \ker(\psi_{\infty,k}) \end{aligned}$$

Since $\bigcup_l F_k^l$ is dense in $A_{i(k)}$, we get $\ker(\omega_k) = \ker(\psi_{\infty,k})$. Then $\ker(\omega) = \overline{\bigcup_k \psi_{\infty,k}(\omega_k)} = \overline{\bigcup_k 0} = 0$, and so ω is injective.

Surjectivity of ω : Let $a \in A$ and $\varepsilon > 0$. We want to check that $a \in_\varepsilon \text{im}(\omega)$. Since the sequence x_1, x_2, \dots is dense in A , there exists some l with $\|a - x_l\| < \varepsilon/4$. Let $k \geq l$ be a number with $1/2^{k-1} < \varepsilon/4$. We have seen above that $\omega_k = \frac{F_k^k}{1/2^{k-1}} \varphi_{i(k)}$. Then:

$$\begin{aligned} a &=_{\varepsilon/4} x_l \\ &\in \{x_1, \dots, x_k\} \\ &\subset_{1/2^k} \varphi_{i(k)}(F_k^k) && [\text{by property (f)}] \\ &\subset_{1/2^{k-1}} \omega_k(F_k^k) && [\text{since } \omega_k = \frac{F_k^k}{1/2^{k-1}} \varphi_{i(k)}] \\ &\subset \text{im}(\omega) \end{aligned}$$

Together, a lies in $\text{im}(\omega)$ up to $\varepsilon/4 + 1/2^i + 1/2^{k-1} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we deduce $a \in \text{im}(\omega)$, and so ω is surjective. \square

3.6. Let us consider a weaker approximation than in 3.5, where we relax condition (A3). So assume, the following situation is given:

Let A be a separable C^* -algebra, and $\{A_i\}_{i \in I}$ a collection of separable C^* -algebras together with morphisms $\varphi_i: A_i \rightarrow A$, such that the following holds: For every index $i \in I$, and $\varepsilon > 0$, and for every finite subsets $F \subset A_i$ and $E \subset \ker(\varphi_i)$, there exists some index j and a morphism $\psi: A_i \rightarrow A_j$ such that:

- (A1) $\varphi_j \circ \psi =_{\varepsilon}^F \varphi_i$
- (A2) $\psi =_{\varepsilon}^E 0$

and moreover, the following condition holds:

- (A3') the sub- C^* -algebras $\text{im}(\varphi_i) = \varphi_i(A_i) \subset A$ approximate A

Adopting the proof of 3.5, we may construct one-sided approximate intertwining to get the following result: For every $\gamma > 0$ and every finite $H \subset A$, there exists a sub- C^* -algebra $B \subset A$ such that $H \subset_\gamma B$ and B is an inductive limit of some of the algebras A_i .

If we denote by $\mathcal{C} = \{A_i \mid i \in I\}$ the class of approximating algebras, then this means precisely that A is AC -like, i.e., A is approximated by sub- C^* -algebras that are inductive limits of algebras in \mathcal{C} .

In general, this does not imply that A is an AC -algebra, i.e., an inductive limit of algebras in \mathcal{C} . In fact, not even a \mathcal{C} -like C^* -algebra need to be an AC -algebra, as can be seen by the following example.

Example 3.7 (see [Dad99]). Let us denote by H the class of (direct sums of) homogeneous C^* -algebras. An inductive limit of C^* -algebras in H is called an AH -algebra. In [Dad99], Dadarlat and Eilers construct a C^* -algebra $A = \varinjlim_k A_k$ that is an inductive limit of AH -algebras A_k (so A is an AAH -algebra) but such that A is not an AH -algebra itself. Thus, in general an AAH -algebra need not be an AC -algebra.

Since quotients of homogeneous algebras are homogeneous again, the C^* -algebra A is also H -like. So the example also shows that in general a \mathcal{C} -like algebra need not be an AC -algebra.

In the example of Dadarlat and Eilers, each $A_k = \varinjlim_n A_k^n$ is an inductive limit of C^* -algebras A_k^n that have the form $\bigoplus_{i=1}^d M_{d_i}(C(X_i))$ with each X_i a three-dimensional CW-complex. It is well-known that $C(X_i)$ is not weakly semiprojective if X_i contains a copy of the two-dimensional disc, see e.g. [ST11, Remark 3.3]. It follows that the algebras A_k^n are not weakly semiprojective, and this the crucial point, as we will see below, 3.9 and 3.12.

Assume \mathcal{C} is a class of weakly semiprojective C^* -algebras, and A is an AC -like C^* -algebra. This means that A has approximating sub- C^* -algebras that have an inductive limit structure. We want to verify the assumption of 3.5, so assume we have a C^* -algebra A_i together with a morphism $\varphi_i: A_i \rightarrow A$. We know that a morphism from a weakly semiprojective C^* -algebra into an inductive limit has approximate lifts, see 2.6.

However, in order to verify (23), we need to twist the original morphisms $\varphi_i: A_i \rightarrow A$ to land in one of the approximating sub- C^* -algebras of A that is an inductive limit. For this, we use the following variant of Loring [Lor97, Lemma 15.2.1, p.118], which was shown to me by Loring and Chigogidze.

Lemma 3.8 (see [Lor97, Lemma 15.2.1, p.118]). *Suppose A is a weakly semiprojective C^* -algebra. Then for every $\varepsilon > 0$, and every finite subset $F \subset A$, there exists $\delta > 0$ and a finite subset $G \subset A$ such that the following holds: Whenever $\varphi: A \rightarrow B$ is a morphism, and $C \subset B$ is a sub- C^* -algebra that contains $\varphi(G)$ up to δ , then there exists a morphism $\psi: A \rightarrow C$ such that $\psi =_\varepsilon^F \varphi$.*

Theorem 3.9 (Criterion for AC -like $\Rightarrow AC$). *Let \mathcal{C} be a class of weakly semiprojective C^* -algebras. Then every separable AC -like C^* -algebra is already an AC -algebra.*

Proof. Assume A is an AC -like C^* -algebra. We want to apply the one-sided approximate intertwining, 3.5, to show that A is an AC -algebra. For this we consider the collection of all morphisms $\varphi: C \rightarrow A$ where C is a C^* -algebra from \mathcal{C} (we may think of this collection as being indexed over $\coprod_{C \in \mathcal{C}} \text{Hom}(C, A)$).

We need to check the requirements for 3.5. So assume the following data is given: A morphism $\varphi: C \rightarrow A$ with $C \in \mathcal{C}$, a tolerance $\varepsilon > 0$, and finite subsets $F \subset C$, $E \subset \ker(\varphi)$ and $H \subset A$. We may assume that F contains E . We need to find a C^* -algebra $C' \in \mathcal{C}$ together with a morphism $\varphi': C' \rightarrow A$, and a morphism $\psi: C \rightarrow C'$ such that (A1), (A2), and (A3) are satisfied.

Applying the above variant of [Lor97, Lemma 15.2.1, p.118], see 3.8, to the weakly semiprojective C^* -algebra C for $\varepsilon/3$ and $F \subset C$, we obtain a $\delta > 0$ and a finite subset $G \subset C$ such that a morphism out of C such that G lands up to δ in a sub- C^* -algebra can be twisted to land in that sub- C^* -algebra while moving the images of F at most by ε . We may assume that $\delta \leq \varepsilon/3$.

Set $H' := H \cup \varphi(G)$, which is a finite subset of A . By assumption, there exists a sub- C^* -algebra $B \subset A$ that contains H' up to δ and which is an AC -algebra, say $B = \varinjlim_k C_k$ with connecting morphisms $\gamma_k: C_k \rightarrow C_{k+1}$. Since $\varphi(G) \subset_\delta B$, there exists a morphism $\alpha: C \rightarrow B$ such that $\varphi =_{\varepsilon/3}^F \alpha$.

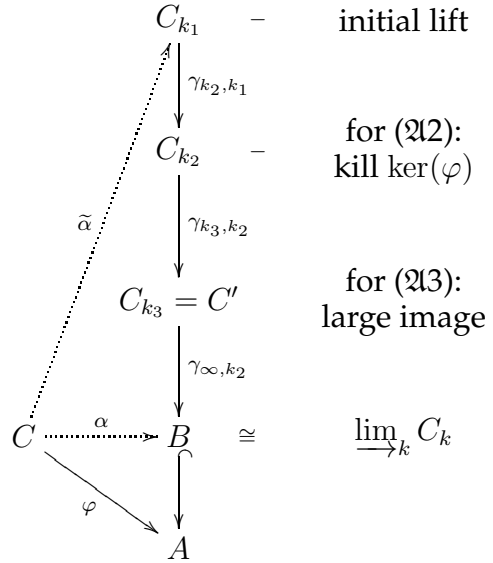
By 2.6, the morphism $\alpha: C \rightarrow B = \varinjlim_k C_k$ has an approximate lift, i.e., there exists an index k_1 and a morphism $\tilde{\alpha}: C \rightarrow C_{k_1}$ such that $\alpha =_{\varepsilon/3}^F \gamma_{\infty, k_1} \circ \tilde{\alpha}$. Then $\varphi =_{\varepsilon/3}^F \gamma_{\infty, k_1} \circ \tilde{\alpha}$. The morphisms are shown in the diagram on the right.

We will now go "further down" the inductive limit to guarantee the properties we need to check.

Step 1 (in order to guarantee (A2)): We consider E . Since $\varphi =_{\varepsilon/3}^F \gamma_{\infty, k_1} \circ \tilde{\alpha}$ and $E \subset F$, we have $\gamma_{\infty, k_1} \circ \tilde{\alpha} =_{\varepsilon/3}^E \varphi =^E 0$. Thus, we may find $k_2 \geq k_1$ such that $\gamma_{k_2, k_1} \circ \tilde{\alpha} =_{\varepsilon}^E 0$.

Step 2 (in order to guarantee (A3)): Since $H \subset_\delta B = \varinjlim_k C_k$, we may find $k_3 \geq k_2$ such that $H \subset_{2\delta} \text{im}(\gamma_{\infty, k_3})$.

Setting $C' := C_{k_3}$, $\varphi' := \gamma_{\infty, k_3}$ and $\psi := \gamma_{k_3, k_1} \circ \tilde{\alpha}: C \rightarrow C' = C_{k_3}$, it is easy to check that (A1), (A2), and (A3) are satisfied. \square



Corollary 3.10 (Criterion for \mathcal{C} -like $\Rightarrow AC$: Loring's local test for inductive limits; see [Lor97, Lemma 15.2.2, p.119]). *Let \mathcal{C} be a class of weakly semiprojective C^* -algebras. Then every \mathcal{C} -like C^* -algebra is an AC -algebra.*

Remark 3.11. Let \mathcal{C} be a class of weakly semiprojective C^* -algebras. If \mathcal{C} is closed under quotients, then every AC -like C^* -algebra is also \mathcal{C} -like, and similarly every AAC -algebra

is \mathcal{C} -like. Then 3.9 and 3.12 follow from Loring's local test for inductive limits, [Lor97, Lemma 15.2.2, p.119], see 3.10.

However, in section 4 we will consider the class \mathcal{P} of projective C^* -algebras, and this class is not closed under quotients. There exist even AP -like C^* -algebras that are not \mathcal{P} -like: Consider for example the commutative C^* -algebra $A = C_0([0, 1]^2 \setminus \{(0, 0)\})$, which is contractible and hence AP -like (even an AP -algebra) by 4.5. Every sub- C^* -algebra of A is commutative, and every commutative projective C^* -algebra has one-dimensional spectrum, as shown by Chigogidze and Dranishnikov, [CD10]. In particular, every commutative projective C^* -algebra has stable rank one, and if A was approximated by such sub- C^* -algebras, then A would have stable rank one as well, which contradicts $sr(A) = 2$.

Therefore, to obtain 4.6 (2) and (3), it is crucial that 3.12 and 3.9 also hold for classes \mathcal{C} that are not necessarily closed under quotients.

Theorem 3.12 (Criterion for $AAC \Rightarrow AC$). *Let \mathcal{C} be a class of weakly semiprojective C^* -algebras. Then every AAC -algebra is already an AC -algebra.*

Proof. Assume $A \cong \varinjlim_k A_k$ and $\varrho_k^n: A_k^n \rightarrow A_k^{n+1}$ for algebras $A_k^n \in \mathcal{C}$. Let us denote the connecting morphisms by $\gamma_k: A_k \rightarrow A_{k+1}$, and $\varrho_k^n: A_k^n \rightarrow A_k^{n+1}$. We are given the following situation:

$$\begin{array}{ccccccc}
 A_k^n & & A_{k+1}^n & & A_{k+2}^n & & \\
 \downarrow \varrho_k^n & & \downarrow \varrho_{k+1}^n & & \downarrow \varrho_{k+2}^n & & \\
 A_k^{n+1} & & A_{k+1}^{n+1} & & A_{k+2}^{n+1} & & \\
 \downarrow \varrho_k^{\infty, n+1} & & \downarrow \varrho_{k+1}^{\infty, n+1} & & \downarrow \varrho_{k+2}^{\infty, n+1} & & \\
 A_k & \xrightarrow{\gamma_k} & A_{k+1} & \xrightarrow{\gamma_{k+1}} & A_{k+2} & \xrightarrow{\quad \cdots \quad} & A \\
 & & & & \searrow \gamma_{\infty, k+2} & &
 \end{array}$$

We want to use the one-sided approximate intertwining 3.5, and we consider the collection of C^* -algebras A_k^n together with morphisms $\varphi_{k,n} := \gamma_{\infty, k} \circ \varrho_k^{\infty, n}: A_k^n \rightarrow A$ (we may think of this collection as being indexed over $\mathbb{N} \times \mathbb{N}$).

Assume some indices k, n are given together with $\varepsilon > 0$, and with finite sets $F \subset A_k^n$, $E \subset \ker(\varphi_{k,n})$ and $H \subset A$. We may assume $E \subset F$. We need to find k', n' and a morphism $\psi: A_k^n \rightarrow A_{k'}^{n'}$ that satisfy (A1) and (A2) and (A3).

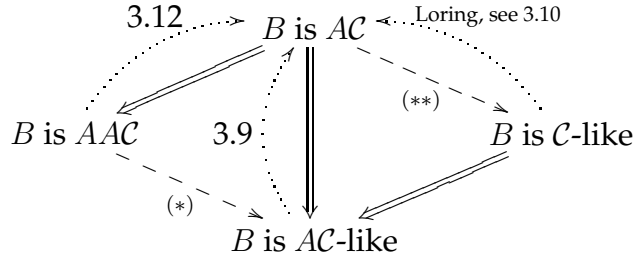
Since $A = \varinjlim_k A_k$ and $\varphi_{k,n} = \gamma_{\infty, k} \circ \varrho_k^{\infty, n} \stackrel{E}{=} 0$, there exists some $k' \geq k$ such that $\gamma_{k', k} \circ \varrho_k^{\infty, n} \stackrel{E}{=} 0$. We may also ensure that $H \subset_{\varepsilon/2} \text{im}(\gamma_{\infty, k'})$, by further increasing k' , if necessary.

Since A_k^n is weakly semiprojective, we may lift the morphism $\gamma_{k', k} \circ \varrho_k^{\infty, n}: A_k^n \rightarrow A_{k'}^{n_1}$ to some $\alpha: A_k^n \rightarrow A_{k'}^{n_1}$ (for some n_1) such that $\varrho_{k'}^{\infty, n_1} \circ \alpha \stackrel{F}{=} \gamma_{k', k} \circ \varrho_k^{\infty, n}$. This is shown in the diagram on the right.

$$\begin{array}{ccccc}
 & & A_{k'}^{n_1} & & \\
 & \nearrow \alpha & \downarrow \varrho_{k'}^{n', n_1} & & \\
 A_k^n & & A_{k'}^{n'} & & \\
 \downarrow \varrho_k^{\infty, n} & & \downarrow \varrho_{k'}^{\infty, n'} & & \\
 A_k & \xrightarrow{\gamma_{k', k}} & A_{k'} & \xrightarrow{\gamma_{\infty, k'}} & A
 \end{array}$$

We have $\varrho_{k'}^{\infty, n_1} \circ \alpha =_{E/\varepsilon/3}^E \gamma_{k', k} \circ \varrho_k^{\infty, n} =_{E/\varepsilon/3}^E 0$. As in the proof of 3.9, we can go "further down" the inductive limit to find $n' \geq n_1$ such that $\varrho_{k'}^{n', n_1} \circ \alpha =_{\varepsilon}^E 0$ and $H \subset_{\varepsilon} \text{im}(\gamma_{\infty, k'} \circ \varrho_{k'}^{\infty, n'})$. Set $\psi := \varrho_{k'}^{n', n_1} \circ \alpha: A_k^n \rightarrow A_{k'}^{n'}$. It is easy to check that $(\mathfrak{A}1)$, $(\mathfrak{A}2)$, and $(\mathfrak{A}3)$ are satisfied. \square

3.13. Let B be a separable C^* -algebra, and \mathcal{C} a class of separable C^* -algebras. The above results give us connections between the four conditions that B is \mathcal{C} -like, or AC -like, or an AC -algebra, or an AAC -algebra. This is shown in the diagram below. A dotted arrow means an implication that holds under the additional assumption that the algebras in \mathcal{C} are weakly semiprojective. The dashed arrow with $(*)$ holds if each quotient of an algebra in \mathcal{C} is an AC -algebra, while the dashed arrow with $(**)$ holds if \mathcal{C} is closed under quotients, see also 3.11.



4. TRIVIAL SHAPE

In this section we study C^* -algebras that are shape equivalent to the zero C^* -algebra 0 . Such algebras are said to have **trivial shape**. We will show in 4.4 that having trivial shape is equivalent to several other natural conditions, most importantly to being an inductive limit of projective C^* -algebras. One may assume that the connecting morphisms are surjective, see 4.9.

We prove some natural closure properties of the class of C^* -algebras with trivial shape, see 4.6. However, building on an example of Dadarlat, [Dad11], see 4.11, we show that trivial shape does not necessarily pass to full hereditary sub- C^* -algebras. It follows that also projectivity does not pass to full hereditary sub- C^* -algebras, see 4.12.

Note that $A \lesssim_{Sh} 0$ implies $A \sim_{Sh} 0$, i.e., A is shape dominated by 0 if and only if it is shape equivalent to 0 . The following recent result of Loring and Shulman was the inspiration for the main result 4.4 below. For the definition of the generation rank $\text{gen}(A)$, see 2.8.

Theorem 4.1 (see [LS10, Theorem 7.4]). *Let A be a C^* -algebra. Then the cone $CA = C_0((0, 1]) \otimes A$ can be written as an inductive limit $CA \cong \varinjlim_k P_k$ of projective C^* -algebras P_k with surjective connecting morphisms $P_k \rightarrow P_{k+1}$, and $\text{gen}(P_k) \leq \text{gen}(A) + 1$.*

Lemma 4.2. *Let $\varphi: A \rightarrow B$ be a projective morphism. Then $\varphi \simeq 0$.*

Proof.

This is a variant of the standard argument for showing that a projective C^* -algebra is contractible. We include it for completeness. Let $\text{ev}_1: CB \rightarrow B$ be the evaluation morphism at 1. The projectivity of φ gives us a lift $\psi: A \rightarrow CB$ such that $\text{ev}_1 \circ \psi = \varphi$. This is indicated in the commutative diagram on the right.

$$\begin{array}{ccc} & & CB \\ & \nearrow \psi & \downarrow \text{ev}_1 \\ A & \xrightarrow{\varphi} & B \end{array}$$

We have $\text{id}_{CB} \simeq 0$ since CB is contractible. Then $\varphi = \text{ev}_1 \circ \text{id}_{CB} \circ \psi \simeq 0$, as desired. \square

Lemma 4.3. *Let (A_k, γ_k) a shape system with inductive limit $A := \varinjlim A_k$. Assume that every semiprojective morphism $D \rightarrow A$ (from any C^* -algebra D) is null-homotopic. Then for each k , there exists $k' \geq k$ such that $\gamma_{k',k} \simeq 0$.*

Proof.

We are given some index k . Note that $\gamma_{k+1,k}$ is semiprojective. Define two morphisms $\sigma_1: A_{k+1} \rightarrow A_{k+2}$ as $\sigma_1 = \gamma_{k+2,k+1}$ and $\sigma_2 = 0$. The morphism $\gamma_{\infty,k+2} \circ \sigma_1 = \gamma_{\infty,k+1}$ is semiprojective, and therefore null-homotopic by assumption. Thus $\gamma_{\infty,k+2} \circ \sigma_1 \simeq 0 = \gamma_{\infty,k+2} \circ \sigma_2$.

$$\begin{array}{ccccc} A_k & \xrightarrow{\gamma_{k+1,k}} & A_{k+1} & & \\ & & \downarrow \sigma_1 \quad \sigma_2 & & \\ & & A_{k+2} & \xrightarrow{\gamma_{k',k+2}} & A_{k'} \xrightarrow{\gamma_{\infty,k'}} A \end{array}$$

Using the semiprojectivity of $\gamma_{k+1,k}$ it follows from [EK86, 3.2], see 2.4, that there exists $k' \geq k+2$ such that $\gamma_{k',k} = \gamma_{k',k+2} \circ \sigma_1 \circ \gamma_{k+1,k} \simeq \gamma_{k',k+2} \circ \sigma_2 \circ \gamma_{k+1,k} = 0$. The situation is shown in the diagram on the right. \square

Theorem 4.4. *Let A be a separable C^* -algebra. Then the following are equivalent:*

- (a) $A \sim_{Sh} 0$
- (b) every semiprojective morphism $D \rightarrow A$ (from any C^* -algebra D) is null-homotopic
- (c) A can be written as $A = \varinjlim A_k$ with projective connecting morphisms $A_k \rightarrow A_{k+1}$
- (d) A can be written as $A = \varinjlim A_k$ with null-homotopic connecting morphisms $A_k \rightarrow A_{k+1}$
- (e) A is an inductive limit of finitely generated, projective C^* -algebras
- (f) A is an inductive limit of finitely generated cones
- (g) A is an inductive limit of contractible C^* -algebras

Moreover, in all decompositions $A \cong \varinjlim_k A_k$ we may assume $\text{gen}(A_k) \leq \text{gen}(A) + 1$.

Proof. Note that 0 has a natural shape system consisting of the zero C^* -algebra at each step. Therefore, $A \sim_{Sh} 0$ means that there exists a shape system (A_k, γ_k) for A and morphisms $\alpha_k: A_k \rightarrow 0$ and $\beta_k: 0 \rightarrow A_{k+1}$ such that $\beta_{k+1} \circ \alpha_k \simeq \gamma_k$. This is shown in the following diagram, which homotopy commutes:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\gamma_1} & A_2 & \xrightarrow{\gamma_2} & A_3 & \longrightarrow & \cdots \longrightarrow A \\ & \searrow \alpha_1 & \nearrow \beta_1 & \searrow \alpha_2 & \nearrow \beta_2 & \searrow \alpha_3 & \\ & 0 & & 0 & & 0 & \longrightarrow 0 \longrightarrow 0 \end{array}$$

"(a) \Rightarrow (d)": Assume $A \sim_{Sh} 0$. We just noted that this implies that A has a shape system A_k with connecting morphisms $\gamma_k: A_k \rightarrow A_{k+1}$ that are null-homotopic since up to homotopy they factor through 0.

"(d) \Rightarrow (a)": Assume there is an inductive system $\mathcal{A} = (A_k, \gamma_k)$ with $A \cong \varinjlim \mathcal{A}$ and null-homotopic connecting morphisms γ_k . Let $\alpha_k: A_k \rightarrow 0$ and $\beta_k: 0 \rightarrow A_{k+1}$ be the zero morphisms. Then $\beta_{k+1} \circ \alpha_k = 0 \simeq \gamma_k$. Conversely also $\beta_k \circ \alpha_k = 0$, so that the inductive systems \mathcal{A} and $(0 \rightarrow 0 \rightarrow \dots)$ are shape equivalent. This does not show $A \sim_{Sh} 0$ right away since the inductive system \mathcal{A} need not be a shape system. However, [Bla85, Theorem 4.8] saves the day: Whenever two inductive systems are shape equivalent, then their inductive limit C^* -algebras are shape equivalent.

"(d) \Rightarrow (b)": Assume there is an inductive system $\mathcal{A} = (A_k, \gamma_k)$ with $A \cong \varinjlim \mathcal{A}$ and null-homotopic connecting morphisms γ_k . Let $\varphi: D \rightarrow A$ be any semiprojective morphism. By [Bla85, Theorem 3.1], see 2.4, there exists k and a morphism $\psi: D \rightarrow A_k$ such that $\varphi \simeq \gamma_{\infty, k} \circ \psi$. But $\gamma_{\infty, k}$ factors as $\gamma_{\infty, k} = \gamma_{\infty, k+1} \circ \gamma_{k+1, k}$ and is therefore null-homotopic. Then φ is null-homotopic as well.

"(b) \Rightarrow (f)": By Blackadar, [Bla85, Theorem 4.3], see 2.3, A has a shape system (A_k, γ_k) with finitely generated algebras A_k and such that $\text{gen}(A_k) \leq \text{gen}(A)$. We may apply 4.3 inductively to this shape system, and after passing to a suitable subsystem we see that there exists a shape system (A_k, γ_k) of finitely generated C^* -algebras A_k with $\text{gen}(A_k) \leq \text{gen}(A)$ and null-homotopic connecting morphisms γ_k such that $A \cong \varinjlim \mathcal{A}$.

A homotopy $\gamma_k \simeq 0$ corresponds naturally to a morphism $\Gamma_k: A_k \rightarrow CA_{k+1}$ such that $\gamma_k = \text{ev}_1 \circ \Gamma_k$, where CA_{k+1} is the cone over A_{k+1} and ev_1 is evaluation at 1.

Set $\omega_k := \Gamma_k \circ \text{ev}_1: CA_k \rightarrow CA_{k+1}$. Consider the inductive system $\mathcal{B} = (CA_k, \omega_k)$. It follows from [LS10, Lemma 7.1] that $\text{gen}(CA_k) \leq \text{gen}(A_k) + 1$, so that CA_k is finitely generated and $\text{gen}(CA_k) \leq \text{gen}(A) + 1$. The systems \mathcal{A} and \mathcal{B} are intertwined, which implies that their inductive limits are isomorphic, so that A is isomorphic to an inductive limit of the finitely generated cones CA_k . The intertwining is shown in the following commutative diagram.

$$\begin{array}{ccccccc}
 & A_1 & \xrightarrow{\gamma_1} & A_2 & \xrightarrow{\gamma_2} & A_3 & \longrightarrow \dots \longrightarrow \varinjlim_k A_k \\
 \text{ev}_1 \nearrow & & & \text{ev}_1 \nearrow & & \text{ev}_1 \nearrow & \\
 & \Gamma_1 \searrow & & \Gamma_2 \searrow & & \Gamma_3 \searrow & \\
 CA_1 & \xrightarrow{\omega_1} & CA_2 & \xrightarrow{\omega_2} & CA_3 & \longrightarrow \dots \longrightarrow \varinjlim_k CA_k
 \end{array}$$

$\varinjlim_k A_k \cong \varinjlim_k CA_k$

"(f) \Rightarrow (e)": Assume $A \cong \varinjlim CA_k$ with each A_k finitely generated and $\text{gen}(A_k) \leq \text{gen}(A)$. By the result of Loring and Shulmann, [LS10, Theorem 7.4], see 4.1, for each k , the cone CA_k can be written as an inductive limit of finitely generated projective C^* -algebras with generator rank at most $\text{gen}(A_k) + 1$. Note that $\text{gen}(A_k) + 1 \leq \text{gen}(A) + 1$ for all k . It follows from 3.12 that A is isomorphic to an inductive limit of finitely generated, projective C^* -algebras with generator rank at most $\text{gen}(A) + 1$.

"(e) \Rightarrow (c)", "(e) \Rightarrow (g)" and "(g) \Rightarrow (d)" are clear. "(c) \Rightarrow (d)" follows from 4.2. \square

Corollary 4.5. *Every separable, contractible C^* -algebra is an inductive limit of projective C^* -algebras.*

Theorem 4.6 (Closure properties of trivial shape). *The class of separable C^* -algebras with trivial shape is closed under:*

- (1) *countable direct sums*
- (2) *inductive limits*
- (3) *approximation by sub- C^* -algebras (i.e., likeness, see 3.2)*
- (4) *taking maximal tensor products with any other (separable) C^* -algebra, i.e., $A \otimes_{\max} B$ has trivial shape as soon as A has trivial shape*

Proof. (1): Assume A_1, A_2, \dots have trivial shape. By condition (g) of the above result 4.4, each A_k can be written as inductive limit of contractible C^* -algebras. Note that countable direct sums of contractible C^* -algebras are again contractible. Hence, $\bigoplus_k A_k$ is an inductive limit of contractible C^* -algebras and thus has trivial shape by (g) of 4.4.

(2): Assume $A \cong \varinjlim A_k$ with each A_k having trivial shape. By 4.4, each A_k is an inductive limit of projective C^* -algebras. It follows from 3.12 that A is an inductive limit of projective C^* -algebras, and so it has trivial shape using 4.4 again.

(3): Assume a C^* -algebra A is approximated by sub- C^* -algebras $A_i \subset A$. By 4.4, each A_i is an inductive limit of projective C^* -algebras. This means that A is \mathcal{AP} -like for the class \mathcal{P} of projective C^* -algebras. It follows from 3.9 that A is an \mathcal{AP} -algebra, i.e., and inductive limit of projective C^* -algebras, and so A has trivial shape by 4.4.

(4): Let A be a C^* -algebra with trivial shape, and B any other (separable) C^* -algebra. By condition (f) of 4.4, we can write A as an inductive limit of cones $CA_k = C_0((0, 1]) \otimes A_k$. As noted by Blackadar, [Bla06, II.9.6.5, p.188], maximal tensor products commute with arbitrary inductive limits (while minimal tensor products only commute with inductive limits with injective connecting morphisms). Thus, $A \otimes_{\max} B$ is the inductive limit of $CA_k \otimes_{\max} B = C_0((0, 1]) \otimes A_k \otimes_{\max} B = C(A_k \otimes B)$. Using condition (f) of 4.4 again, we deduce that $A \otimes_{\max} B$ has trivial shape. \square

Corollary 4.7 (Contractible-like C^* -algebras have trivial shape). *Let A be a separable C^* -algebra that is approximated by contractible sub- C^* -algebras. Then A has trivial shape.*

Corollary 4.8 (Contractible-like C^* -algebras are inductive limits of contractibles). *Let A be a separable C^* -algebra that is approximated by contractible sub- C^* -algebras. Then A is an inductive limit of contractible C^* -algebras.*

Proposition 4.9. *Let (A_k, γ_k) be an inductive system of separable C^* -algebras. Then there exists an inductive system (B_k, δ_K) with surjective connecting morphisms and such that $\varinjlim A_k \cong \varinjlim B_k$. Moreover, we may assume $B_k = A_k * \mathcal{F}_\infty$ (the free product), where*

$$\mathcal{F}_\infty := C^*(x_1, x_2, \dots \mid \|x_i\| \leq 1)$$

is the universal C^ -algebra generated by a countable number of contractive generators. If A_k is (semi-)projective, then so is $A_k * \mathcal{F}_\infty$.*

Proof. The algebras A_k are separable. Thus, for each k there exists a surjective morphism $\varphi_k: \mathcal{F}_\infty \rightarrow A_k$, sending the generator x_j to $\varphi_k(x_j) \in A_k$. Consider the universal C^* -algebra

$$\mathcal{G} := C^*(x_{i,j} \mid i, j \in \mathbb{N}, \|x_{i,j}\| \leq 1).$$

The only difference from \mathcal{F}_∞ is the other enumeration of generators. Using a bijection $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, we may construct an isomorphism $\mathcal{G} \cong \mathcal{F}_\infty$.

Set $B_k := A_k * \mathcal{G}$ and define a morphism $\psi_k: \mathcal{G} \rightarrow B_{k+1}$ by defining its value on the generators as $\psi_k(x_{1,j}) := \varphi_{k+1}(x_j)$, and $\psi_k(x_{i,j}) := x_{i-1,j}$ if $i \geq 2$. Define a morphism $\delta_k: B_k \rightarrow B_{k+1}$ as $\delta_k := \gamma_k * \psi_k$. It is easy to check that δ_k is surjective.

For each i , the elements $x_{i,1}, x_{i,2}, \dots \in \mathcal{G}$ generate a copy of \mathcal{F}_∞ . In this way, we may think of \mathcal{G} as a countable free product of copies of \mathcal{F}_∞ . Then, the map δ_k looks as follows:

$$\begin{array}{ccccccc}
 B_k & := & A_k & * & \mathcal{F}_\infty & * & \mathcal{F}_\infty & * & \mathcal{F}_\infty & * & \dots \\
 \delta_k \downarrow & & \gamma_k \downarrow & \swarrow \varphi_k & \searrow \cong & \searrow \cong & \searrow \cong & & & & \\
 B_{k+1} & := & A_{k+1} & * & \mathcal{F}_\infty & * & \mathcal{F}_\infty & * & \mathcal{F}_\infty & * & \dots
 \end{array}$$

Consider the natural inclusions $\iota_k: A_k \rightarrow B_k$. These intertwine with the connecting morphisms γ_k and δ_k , i.e., $\delta_k \circ \iota_k = \iota_{k+1} \circ \gamma_k$. Thus, the morphisms ι_k define a natural morphism $\iota: A = \varinjlim A_k \rightarrow B := \varinjlim B_k$. Since each ι_k is injective, so is ι .

Let us check that ι is also surjective. Let $b \in B$ and $\varepsilon > 0$ be given. We need to find some $a \in A$ with $b =_\varepsilon \iota(a)$, i.e., $\|b - \iota(a)\| < \varepsilon$. First, we may find an index k and $b' \in B_k$ such that $\delta_{\infty,k}(b') =_{\varepsilon/2} b$. By definition, $B_k = A_k * \mathcal{G}$. This implies that every element of B_k can be approximated by finite polynomials involving the elements of A and the generators $x_{i,j}$. Actually, we only need that b' is approximated up to $\varepsilon/2$ by an element b'' in the sub- C^* -algebra $A_k * C^*(x_{i,j} \mid i \in \{1, 2, \dots, l\}, j \in \mathbb{N}, \|x_{i,j}\| \leq 1)$. Note that $\delta_{k+l,k}(b'')$ lies in the image of ι_{k+l} , say $\delta_{k+l,k}(b'') = \iota_{k+l}(x)$ for $x \in A_{k+l}$. Then $a = \gamma_{\infty,k+l}(x) \in A$ satisfies $b =_\varepsilon \iota(a)$, which completes the proof of surjectivity.

Note that \mathcal{F}_∞ is projective. It follows from [Bla85, Proposition 2.6, 2.31] that $A_k * \mathcal{F}_\infty$ is (semi-)projective, if A_k is so. \square

Corollary 4.10. *If a separable C^* -algebra has trivial shape, then it is an inductive limit of projective C^* -algebra with surjective connecting morphisms.*

Remark 4.11 (see [Dad11]). Dadarlat gives an example of a commutative C^* -algebra $A = C_0(X, x_0)$ such that $A \otimes \mathbb{K}$ is contractible (in particular has trivial shape), while A is not contractible. In fact, X is a two-dimensional CW-complex with non-trivial fundamental group, so that (X, x_0) does not have trivial shape (in the pointed, commutative category). It follows from [Bla85, Proposition 2.9] that $C_0(X, x_0)$ also does not have trivial shape (as a C^* -algebra).

Thus, while $A \otimes \mathbb{K}$ has trivial shape, the full hereditary sub- C^* -algebra $A \subset A \otimes \mathbb{K}$ does not. This shows that trivial shape does not pass to full hereditary sub- C^* -algebras. From this we may deduce the following result.

Proposition 4.12. *Projectivity does not pass to full hereditary sub- C^* -algebras.*

Proof. Let A be Dadarlat's example of a C^* -algebra with $A \otimes \mathbb{K} \simeq 0$ while $A \approx_{Sh} 0$, see [Dad11] and 4.11. By 4.10, $A \otimes \mathbb{K}$ is an inductive limit of projective C^* -algebra P_k with surjective connecting morphisms $\gamma_k: P_k \rightarrow P_{k+1}$. Consider the pre-images $Q_k := \gamma_{\infty,k}^{-1}(A) \subset P_k$. Since $A \subset A \otimes \mathbb{K}$ is a full hereditary sub- C^* -algebra, so is $Q_k \subset P_k$.

Note that $A \cong \varinjlim Q_k$. If all algebras Q_k were projective, then A would have trivial shape by 4.4. Since this is not the case, some algebras Q_k are not projective. \square

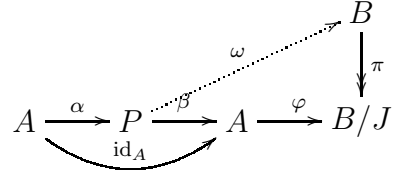
5. RELATIONS AMONG THE CLASSES OF (WEAKLY) (SEMI-)PROJECTIVE C^* -ALGEBRAS

In this section we will study the relation among the four classes of (weakly) (semi-)projective C^* -algebras. As it turns out, the situation is completely analogous to the commutative setting.

Lemma 5.1. *Let A be a C^* -algebra, P a projective C^* -algebra and $\alpha: A \rightarrow P$, $\beta: P \rightarrow A$ two morphisms with $\beta \circ \alpha = \text{id}_A$. Then A is projective.*

Proof. Let B be any C^* -algebra, $J \triangleleft B$ an ideal, and $\varphi: A \rightarrow B/J$ a morphism. We need to find a lift $\psi: A \rightarrow B$.

Since P is projective, there exists a morphism $\omega: P \rightarrow B$ that lifts $\varphi \circ \beta: P \rightarrow B/J$, i.e., $\pi \circ \omega = \varphi \circ \beta$. Set $\psi := \omega \circ \alpha: A \rightarrow B$. Then $\pi \circ \psi = \pi \circ \omega \circ \alpha = \varphi \circ \beta \circ \alpha = \varphi \circ \text{id}_A$. The situation is shown in the diagram on the right.

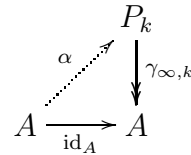


\square

Theorem 5.2. *Let A be a semiprojective C^* -algebra of trivial shape. Then A is projective.*

Proof. By 4.10, A is an inductive limit of projective C^* -algebra P_k with surjective connecting morphisms $\gamma_k: P_k \rightarrow P_{k+1}$.

The semiprojectivity of A gives an index k and a lift $\alpha: A \rightarrow P_k$ such that $\gamma_{\infty,k} \circ \alpha = \text{id}_A$. From the above lemma 5.1, we get that A is projective. The situation is shown in the diagram on the right.



\square

Since every projective C^* -algebra is contractible, we get the following corollary:

Corollary 5.3. *Let A be a semiprojective C^* -algebra of trivial shape. Then A is contractible.*

Loring, [Lor09, Lemma 5.5], shows that for a weakly projective C^* -algebra A and a semiprojective C^* -algebra D the set $[D, A]$ of homotopy classes of morphisms from D to A is trivial. A variant of this proof shows condition (b) in 4.4, so that we get the following:

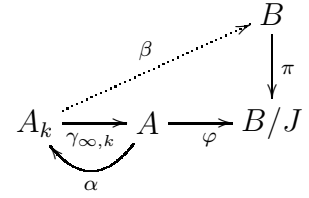
Proposition 5.4 (see [Lor09, Lemma 5.5]). *Every weakly projective C^* -algebra has trivial shape.*

This result of Loring shows that a weakly projective C^* -algebra is weakly semiprojective and has trivial shape. We will now show that the converse is also true.

Theorem 5.5. *Let A be a weakly semiprojective C^* -algebra of trivial shape. Then A is weakly projective.*

Proof. Let B be a C^* -algebra, let $J \triangleleft B$ be an ideal, and $\pi: B \rightarrow B/J$ the quotient morphism. Let $\varphi: A \rightarrow B/J$ be a morphism. Let $F \subset A$ be a finite set, and $\varepsilon > 0$. We need to find a lift $\psi: A \rightarrow B$ such that $\pi \circ \psi =_F^\varepsilon \varphi$.

From 4.4 we get an inductive system (P_k, γ_k) of projective C^* -algebras P_k with inductive limit A . Considering the identity morphism $\text{id}_A: A \rightarrow A \cong \varinjlim P_k$ we get from 2.6 an index k and a morphism $\alpha: A \rightarrow P_k$ such that $\gamma_{\infty,k} \circ \alpha =_F^\varepsilon \text{id}_A$. Consider the morphism $\varphi \circ \gamma_{\infty,k}: P_k \rightarrow B/J$. The projectivity of P_k gives us a lift $\beta: P_k \rightarrow B$ such that $\pi \circ \beta = \varphi \circ \gamma_{\infty,k}$. The situation is shown in the diagram on the right.



Set $\psi := \beta \circ \alpha$. Then $\pi \circ \psi = \pi \circ \beta \circ \alpha = \varphi \circ \gamma_{\infty,k} \circ \alpha =_F^\varepsilon \varphi$ as desired. \square

We summarize the results as follows:

Theorem 5.6. *Let A be a C^* -algebra. Then the following are equivalent:*

- (a) A is (weakly) projective
- (b) A is (weakly) semiprojective and has trivial shape

Corollary 5.7. *Let A be a C^* -algebra. Then the following are equivalent:*

- (a) A is projective
- (b) A is semiprojective and weakly projective
- (c) A is semiprojective and contractible
- (d) A is semiprojective and has trivial shape

5.8. The above result 5.7 confirms a conjecture of Loring. We note that these results are in exact analogy to results in commutative shape theory.

A (weakly) projective C^* -algebra is the non-commutative analogue of an (approximate) absolute retract, and a (weakly) semiprojective C^* -algebra is the non-commutative analogue of an (approximate) absolute neighborhood retract (see [ST11, 2.1, 2.2, 2.3] and the references therein for definitions and further discussion).

With the obvious abbreviations we get the following picture how the classes of (weakly) (semi-)projective C^* -algebras relate, analogously to the classes of (approximate) absolute (neighborhood) retracts.

commutative world (for compact, metric space X):	Reference	noncommutative world (for separable C^* -algebra A):	Reference
• X is AR $\Leftrightarrow X$ is ANR and $X \simeq \text{pt}$	[Bor67, IV.9.1]	• A is P $\Leftrightarrow A$ is SP and $A \simeq 0$	5.2
• X is AAR $\Leftrightarrow X$ is AANR and $X \sim_{Sh} \text{pt}$	[Gmu71], [Bog75]	• A is WP $\Leftrightarrow A$ is WSP and $A \sim_{Sh} 0$	[Lor09], 5.5
• if X is ANR, then: $X \sim_{Sh} \text{pt} \Leftrightarrow X \simeq \text{pt}$	[Bor67]	• if A is SP, then: $A \sim_{Sh} 0 \Leftrightarrow A \simeq 0$	5.3

6. QUESTIONS

Question 6.1. Assume A has trivial shape. Can we write A as an inductive limit $A \cong \varinjlim A_k$ with surjective connecting morphisms of projective C^* -algebras A_k with $\text{gen}(A_k) \leq \text{gen}(A) + 1$?

The result of Loring and Shulmann, [LS10, Theorem 7.4], shows that this is possible for cones. Also, it follows from 4.4 that A is an inductive limit of projective C^* -algebras A_k with $\text{gen}(A_k) \leq \text{gen}(A) + 1$, but the connecting morphisms may not be surjective. Using 4.9, we can always arrange for surjective connecting morphisms, but the approximating algebras are replaced by $A_k * \mathcal{F}_\infty$ which have $\text{gen}(A_k * \mathcal{F}_\infty) = \infty$.

Question 6.2. Say A has property $(*)$ if $[D, A]$ is trivial for every semiprojective C^* -algebra D . Every C^* -algebra of trivial shape has property $(*)$. What about the converse?

If a C^* -algebra A is an inductive limit of semiprojective C^* -algebras, then property $(*)$ for A implies that A has trivial shape. It is however an open question whether every C^* -algebra has such an inductive limit decomposition.

ACKNOWLEDGMENTS

I thank Eduard Ortega for his valuable comments on a first version of this paper, and especially for his careful reading of the technical proofs in part 3. I thank Tatiana Shulman and Leonel Robert for discussions and feedback on this paper.

I thank Mikael Rørdam and George Elliott for interesting discussions on approximate intertwining.

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